3.1.1 Derivative of a Function: The Definition

Recall: Instantaneous Rate of Change and the Slope of the Tangent Line:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Definition of the Derivative: The derivative of *f* at *x* is defined as:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Notation: f', y', $f'(x)$, $D_x[y]$, $\frac{d}{dx}[f(x)]$

Example 1: Applying the Definition

Differentiate (that is, find the derivative of):

1. $f(x)=x^3$

$f(x)=x^2$

$f'(x)=2x$

x^8

$8x^7$

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$\lim_{h\to 0}\frac{\cancel{x^3}+3h^2x+3hx^2+h^3\cancel{-x^3}}{h}$$

$$\lim_{h\to 0}\frac{\cancel{h}(3hx+3x^2+h^2)}{\cancel{h}}$$

$$3(0)x+3x^2+(0)^2$$

$$f'(x)=3x^2$$

$(x+h)(x+h)(x+h)$

$x^2+2hx+h^2$

$x^3+3h^2x+3hx^2+h^3$

1
1 1
1 2 1
1 3 3 1

2. $f(x)=\sqrt{x+1}$

$$\lim_{h\to 0}\frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)}{h}\cdot\frac{\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

THINK

$f(x)=(x+1)^{\frac{1}{2}}$

$$\lim_{h\to 0}\frac{\cancel{x}+h+\cancel{1}-(\cancel{x}+\cancel{1})}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$f'(x)=\frac{1}{2(x+1)^{\frac{1}{2}}}$

$=\frac{1}{2}(x+1)^{-\frac{1}{2}}$

$$\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$f'(x)=\frac{1}{\sqrt{x+(0)+1}+\sqrt{x+1}}$$

$$f'(x)=\frac{1}{2\sqrt{x+1}}$$

3. $f(x)=\frac{1}{x}$

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{(x+h)}{x(x+h)}}{h}$$

think:

$f(x)=x^{-1}$

$$\lim_{h\to 0}\frac{\cancel{x}-\cancel{x}-h}{hx(x+h)}$$

$f'(x)=-x^{-2}$

$$\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$f'(x)=-\frac{1}{x(x+0)}$$

$$=-\frac{1}{x^2}$$

Alternate Definition of the Derivative at a Point

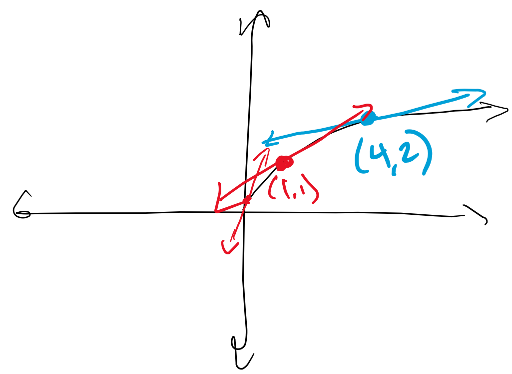
useful for direct substitution of values

at a point $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Example 2: Applying the Alternate Definition

Differentiate $f(x)=\sqrt{x}$ using the alternate definition.



$$f'(a) = \lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})}$$

$$= \lim_{x \to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$$

$$= \lim_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}}$$

$$= \frac{1}{\sqrt{a}+\sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

$$\frac{1}{2\sqrt{1}}$$

$$\frac{1}{2}$$

$a=4$

at the point $(4,2)$ $\quad f(x)=\sqrt{x}$

- evaluate the derivative
- find the slope of the tangent
- find the instantaneous rate of change

$$f'(a) = \frac{1}{2\sqrt{a}}$$

$$f'(4) = \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{4}$$

$$\lim_{x \to 4} \frac{(\sqrt{x}-\sqrt{4})}{x-4} \cdot \frac{(\sqrt{x}+\sqrt{4})}{(\sqrt{x}+\sqrt{4})}$$

$$\lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+\sqrt{4})}$$

$$\frac{1}{\sqrt{4}+\sqrt{4}}$$

$$\frac{1}{4}$$

Example 3: Expressing the derivative as a limit

Express the derivative of each of the following functions as a limit.

a) $y = \cos x$ $\qquad \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

b) $y = 3^x$ $\qquad \lim_{h \to 0} \frac{3^{(x+h)} - 3^x}{h}$

c) $y = 3$ $\qquad f(x+h)$ $\qquad \lim_{h \to 0} \frac{3 - 3}{h} \Rightarrow 0$

Example 4: Recognizing a given limit as a derivative

Given $f'(x)$, determine $f(x)$.

a) $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$ $\qquad f(x) = x^2$

b) $\lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$ $\qquad f(x) = \sqrt[3]{x}$

c) $\lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$ $\qquad f(x) = \tan x$

d) $\lim_{h \to 0} \frac{\frac{x+h}{(x+h)^2 + 1} - \frac{x}{x^2 + 1}}{h}$ $\qquad f(x) = \frac{x}{x^2+1}$

e) $\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$ $\qquad f(x) = e^x$

f) $\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$ $\qquad f(x) = \frac{1}{x}$

Assignment 3.1.1:
Page 101 – 102 #1, 2, 4-6, 11, 12 & Recognizing the Limit as a Derivative Sheet